Exploring with Giants

Michael Abbott

featuring work with
INES ANICETO and OLOF OHLSSON SAX
published in 0811.2423 and 0903.3365

Semi-classical Approaches to Quantum Gravity and String Theory Solitons

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1. A reminder about AdS/CFT
   Large-J limits, spin chains
2. Giant magnons
   Pohlmeyer's map, finite *J*
3. ABJM theory
   Strings in AdS_4 x CP^3
4. Giant magnons in the sigma-model
   Recycled and new solutions
5. The algebraic curve
   Finite-*J* corrections

1. A reminder about AdS/CFT
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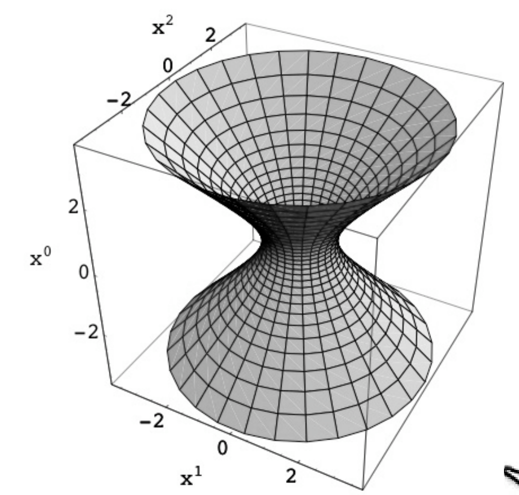
5. The algebraic curve
Finite-*J* corrections

Not in this talk
(but in my thesis)
is work on:

de Sitter space,

Yang black holes,

T-dual magnons.



1. AdS/CFT

[Maldacena, 1997]

(in the best-studied example)
says that

- **IIB superstrings on AdS_5 x S^5**

and

- **Planar *N*=4 SYM**

are secretly the same theory...

... just written in variables which
we understand in opposite limits.

N=4 Super-Yang-Mills theory

$$S_{YM} = \frac{2}{g_{YM}^2}\int d^4x\,\mathrm{Tr}\left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D_\mu\phi_i D^\mu\phi_i - \tfrac{1}{4}[\phi_i,\phi_j][\phi_i,\phi_j] + \tfrac{1}{2}\overline{\chi}\not{D}\chi - \frac{i}{2}\bar{\chi}\Gamma_i[\phi_i,\chi]\right)$$

Symmetries:

- *SU*(N) gauge symmetry. All fields in the adjoint.
- *so*(2,4) conformal symmetry (including Δ).
- *so*(6) = *su*(4) R-symmetry. Scalars in the **6**.

} + fermionic generators = *psu*(2,2|4).

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Symmetries:

- ❖ SU(N) gauge symmetry. All fields in the adjoint.
- ❖ so(2,4) conformal symmetry (including Δ).
- ❖ so(6) = su(4) R-symmetry. Scalars in the **6**.

} + fermionic generators = psu(2,2|4).

What we're interested in calculating is the conformal dimensions Δ of operators:

$$\langle \mathcal{O}_A(x) \mathcal{O}_B(y) \rangle = \frac{\delta_{AB}}{(x-y)^{2\Delta(\mathcal{O}_A)}}$$

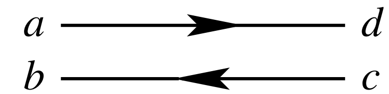
The path integral is:

$$Z = \exp \left(\frac{i}{\hbar} \frac{1}{g_{YM}^2} \int d^4x \mathcal{L} \right)$$

so g_{YM}^2 plays the same role as $\hbar$.

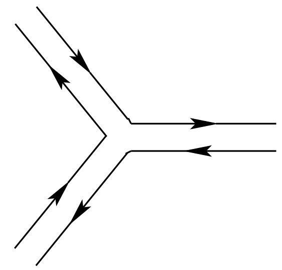
Planar? Consider just the scalars:
Each propagator is:

$$\left\langle \phi_{i\,ab}(x)\phi_{j\,cd}(y)\right\rangle = \frac{g_{YM}^2}{8\pi(x-y)^2}\delta_{ij}\delta_{ad}\delta_{bc} =$$



$$\sim g_{YM}^2.$$

and the vertices are:



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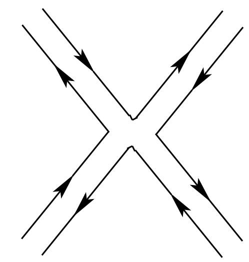

$$\sim \frac{1}{g_{YM}^2}.$$

so altogether, a given diagram will go as $N^F\left(g_{YM}^2\right)^{E-V}$

$$= N^{V-E+F}\left(g_{YM}^2 N\right)^{E-V} = N^{\chi}\lambda^{E-V}$$

Planar? Consider just the scalars:
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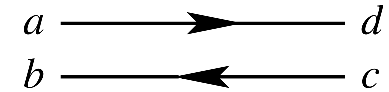

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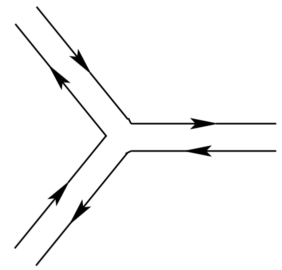

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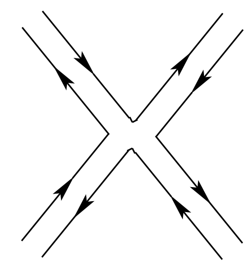

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so altogether, a given diagram will go as $N^F\left(g^2_{YM}\right)^{E-V}$

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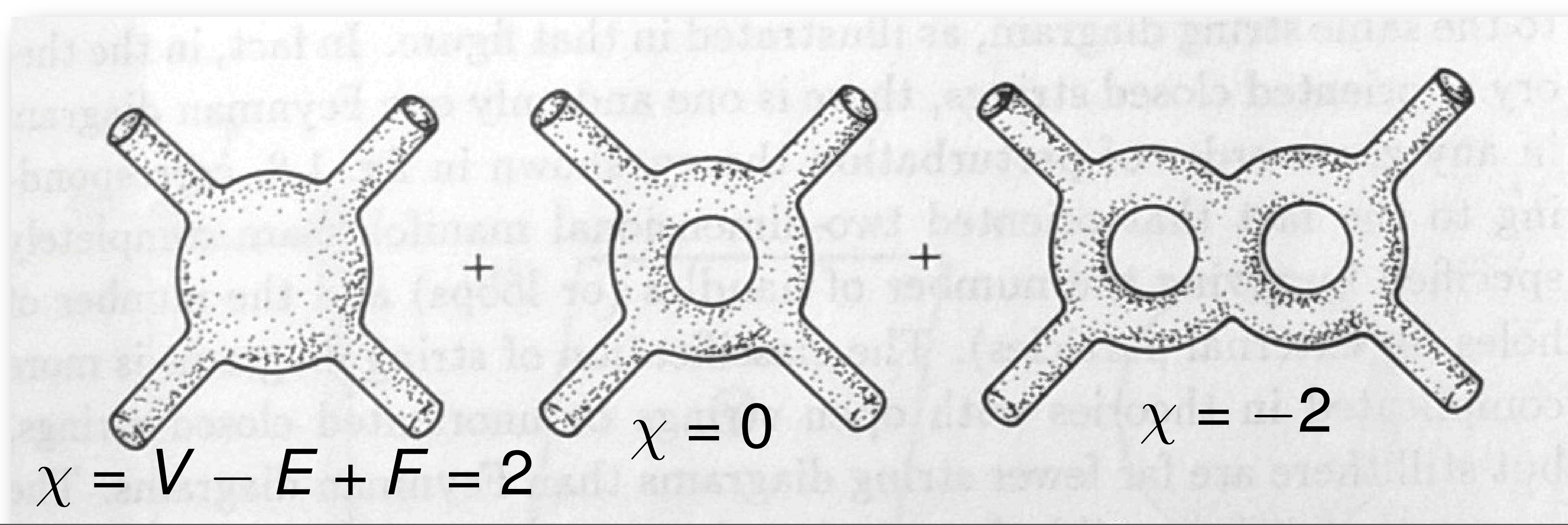
The idea is: we can use $1/N^2$ as an alternative expansion parameter.
This counts genus, just as in string theory.
At large N, the effective field theory coupling is $\lambda = g^2_{YM}N$.

['t Hooft, 1974]



Strings in $\mathbf{AdS_5} \times \mathbf{S^5}$

The bosonic action (in conformal gauge) reads:

$$S = \frac{R^2}{4\pi\alpha'} \int dxdt \, \frac{1}{2} \left(\partial_a \mathbf{Y} \cdot \partial^a \mathbf{Y} - \Lambda(\mathbf{Y}^2 + 1) + \partial_a \mathbf{X} \cdot \partial^a \mathbf{X} - \Lambda'(\mathbf{X} \cdot \mathbf{X} - 1) \right)$$

- Groups SO(2,4) and SO(6) describe rotations of $\mathbb{R}^{2,4}$ and $\mathbb{R}^6$.

  In fact $AdS_5 \times S^5$ is the bosonic part of $\dfrac{PSU(2,2|4)}{SO(4,1) \times SO(5)}$.
- Notice that only $R/\sqrt{\alpha'}$ appears, and $R \gg \sqrt{\alpha'}$ is the classical limit:

$$Z = \int DX \ldots \exp\left(\frac{iR^2}{\hbar\, \alpha'} \int d^2x \mathcal{L} \right)$$

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We'll mostly work in $\mathbb{R} \times S^5$ with $t = \tau_{AdS}$,
and will need charges

$$J = \frac{\sqrt{\lambda}}{2\pi} \int dx \; (X_1 \partial_t X_2 - X_2 \partial_t X_1) \quad \text{and } Q \text{ sim. with } X_3 \,\&\, X_4$$

$$\Delta = \frac{\sqrt{\lambda}}{2\pi} \int dx \, 1$$

AdS/CFT says that the parameters of the two theories are related

$$\left(\frac{R}{\sqrt{\alpha'}}\right)^4 = \lambda = g_{YM}^2 N$$

$$4\pi g_{\mathrm{string}} = g_{YM}^2$$

Perturbative YM: $1 + \lambda + \lambda^2 + \ldots$

Semiclassical strings: $1 + 1/\sqrt{\lambda} + 1/\lambda + \ldots$

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Large-J Limits

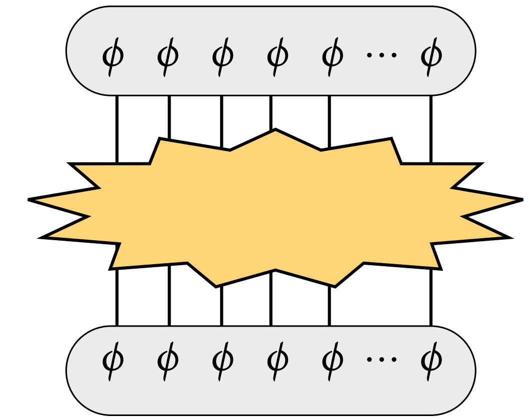
One view is that large N is a made-up parameter (N = 3 for QCD) to help us at strong coupling.

[Berenstein, Maldacena, Nastase 2002]

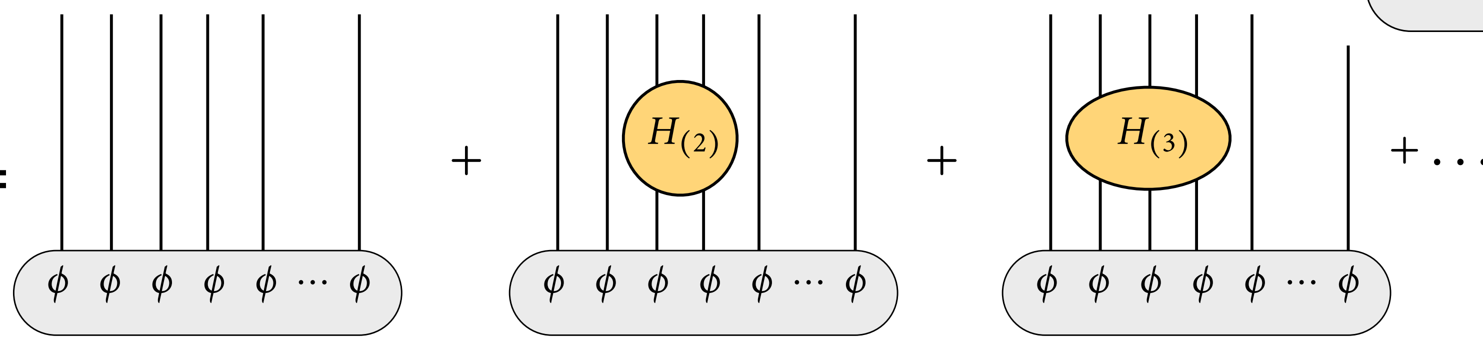
The idea of BMN was to smuggle in one more parameter: let one of the *so*(6) charges, J, become very large.

In their original limit, this allowed them to access both sides: effective gauge coupling $\lambda' = \lambda/J^2$ small while λ large.
We'll instead keep J and independent.

Spin Chains

When working out Δ, we evaluate diagrams $\langle \mathcal{O}, \mathcal{O} \rangle =$ 

whose (planar!) loop expansion reads:

$\langle \mathcal{O}, \mathcal{O} \rangle =$ 

In the $SU(2)$ sector, one-loop Δ-J is the Heisenberg spin chain Hamiltonian:

[Minahan & Zarembo, 2002]

$$H = \sum_i \mathcal{H}_i = \frac{\lambda}{8\pi^2} \sum_i \mathbf{s}_i \cdot \mathbf{s}_{i+1}$$

as long as we interpret $\mathrm{Tr}(ZZZZXZZXZZZ) = \downarrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow$.

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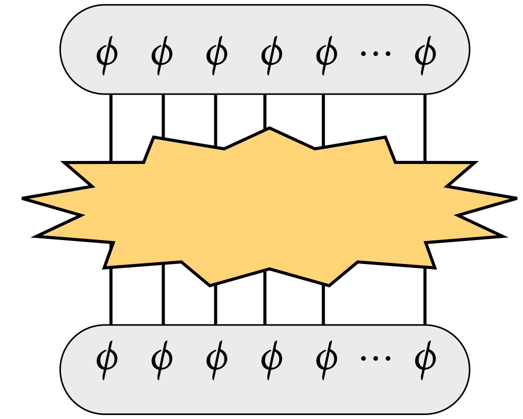

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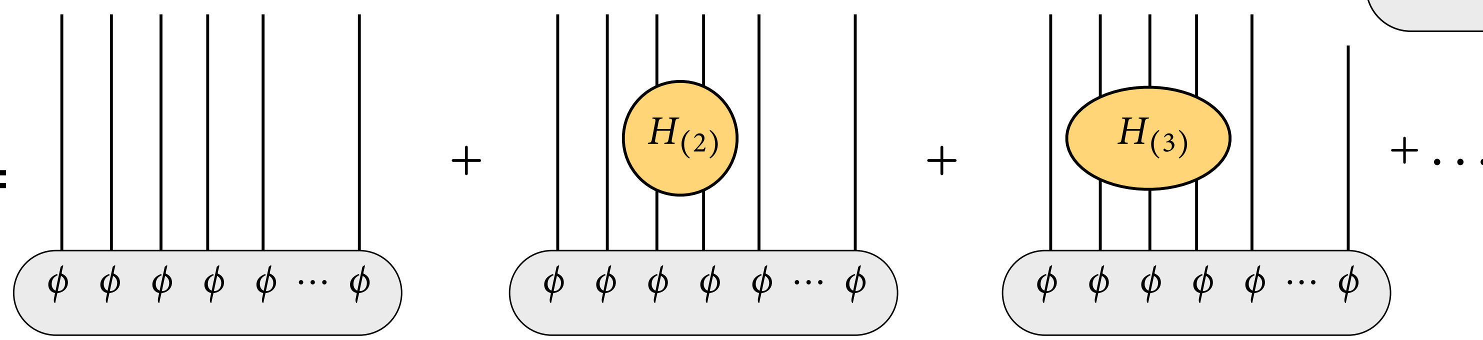

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as long as we interpret Tr(ZZZZXZZXZZZ) = $\downarrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\downarrow$.

And this system has spin-flip excitations called magnons, which have

$$\boxed{\mathcal{E} = \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{p}{2}\right)}$$

Larger su(2|3) sector with length-changing operators: derive from symmetry.

[Inês, friday morning] [Beisert, 2004]

The earlier BMN operators had $p \sim 1/J \ll 1$, here we allow this to be order 1.

The Bethe Ansatz

From these magnons, one can exactly solve the system.

Write one magnon as $|p\rangle = \sum_x e^{ipx} \, |x\rangle$
then his anstaz for two-magnon energy eigenstates was

$$|p_1, p_2\rangle = \sum_{x_1, x_2} \left(e^{ip_1 x_1 + ip_2 x_2} + S(p_1, p_2) e^{ip_1 x_2 + ip_2 x_1} \right) |x_1, x_2\rangle$$

where $S(p_1,p_2)$ is the two-particle S-matrix. [Bethe, 1931]

You can go on to diagonalise the n-magnon sector using the fact that the S-matrix factorises: this is an integrable system.

(Analytic BA: [Faddeev et al.])

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Extensions

This is a completely trivial sector of AdS/CFT. Try to extend...

Full theory, four loops. [Beisert 2004] & references!

All-loop conjectures. Gromov & Vieira, Arutynov, Frolov, Staudacher, ...

2. Giant Magnons

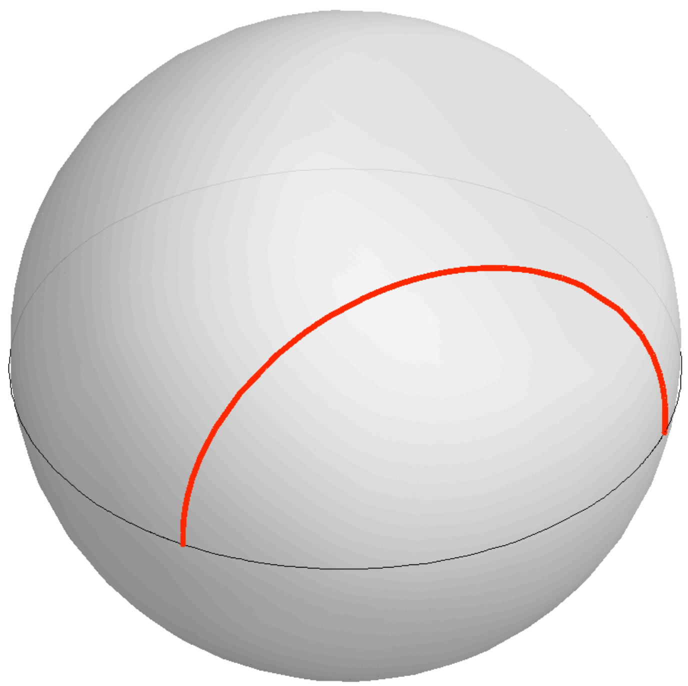
Are the string solutions dual to spin chain excitations, with finite *p*.

Explicitly:

$$X^0 = t$$

$$X^1 + iX^2 = e^{it}\left(v + i\sqrt{1-v^2}\tanh u\right)$$

$$X^3 = \sqrt{1-v^2}\,\mathrm{sech}\,u.$$



[Hofman & Maldacena, 2006]

where $u = \dfrac{x - v\,t}{\sqrt{1-v^2}}$.

and the worldsheet velocity is

$$v = \cos(p/2)$$

The dispersion relation is

$$\boxed{\Delta - J = \frac{\sqrt{\lambda}}{\pi}\sin\left(\frac{p}{2}\right)}$$

The periodic *p* is now the opening angle along the equator.

We need several magnons to make up a closed string:

$$\sum_i p_i = 0 \bmod 2\pi$$

Dyonic Giant Magnons

are solutions with a second large angular momentum $Q \sim \sqrt{\lambda}$

dual to a bound state of Q impurities.

$$\boxed{\Delta - J = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}}$$

[Dorey 2006]

[Chen, Dorey, Okamura, 2006]

These live in S^3,

$$X^1 + iX^2 = e^{it} \cos\theta_{\mathrm{d}}(u)$$

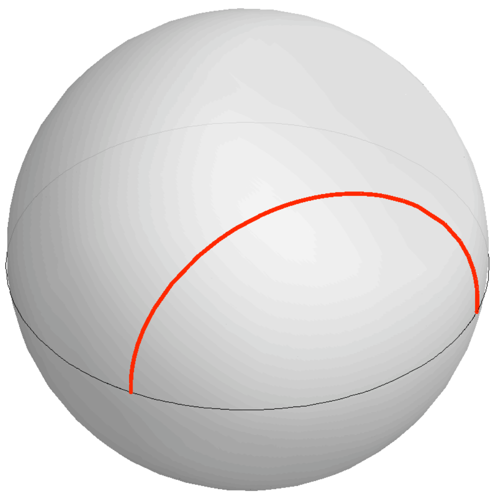
$$X^3 + iX^4 = e^{i\omega t} \sin\theta_{\mathrm{d}}(u)$$

Metric $ds^2 = d\theta^2 + \sin^2\theta \; d\phi^2 + \cos^2\theta \; d\psi^2$,
ansatz $\phi = t$, $\psi = \omega t$ and $\theta = \theta(u)$.

(Actually found using complex sine-gordon.)

We'll discuss two other points of view:

String sigma-model



[Pohlmeyer, 1976]

Map is

$$\cos\alpha = \partial_+ \mathbf{X} \cdot \partial_- \mathbf{X}.$$

Giant Magnon is sent to the kink:

$$\alpha = -4\arctan(e^{-u})$$

with energy

$$E_{s.g} = 8\gamma = \frac{8}{\sin(p/2)}$$

Sine-gordon theory

Worldsheet manifest,
but charges opaque.

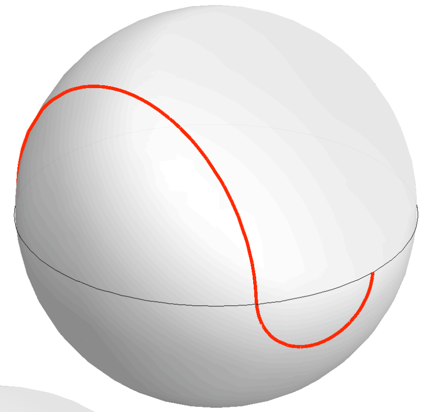
Algebraic Curves

Worldsheet integrated over,
but charges clear.

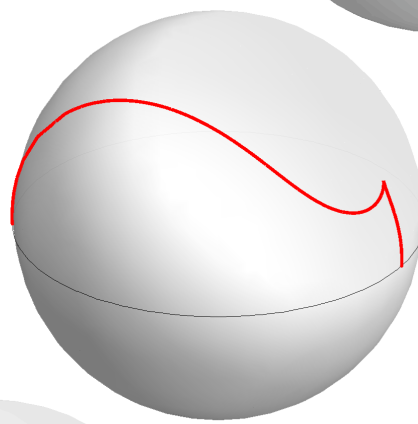
Larger sectors:
[Grigoriev & Tseytlin]
[Mikhailov & Schafer-Nameki]

Scattering Magnons

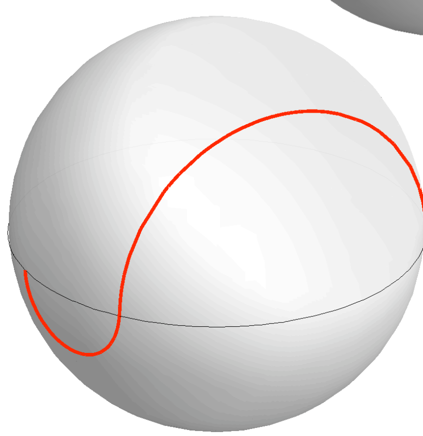
on the worldsheet.



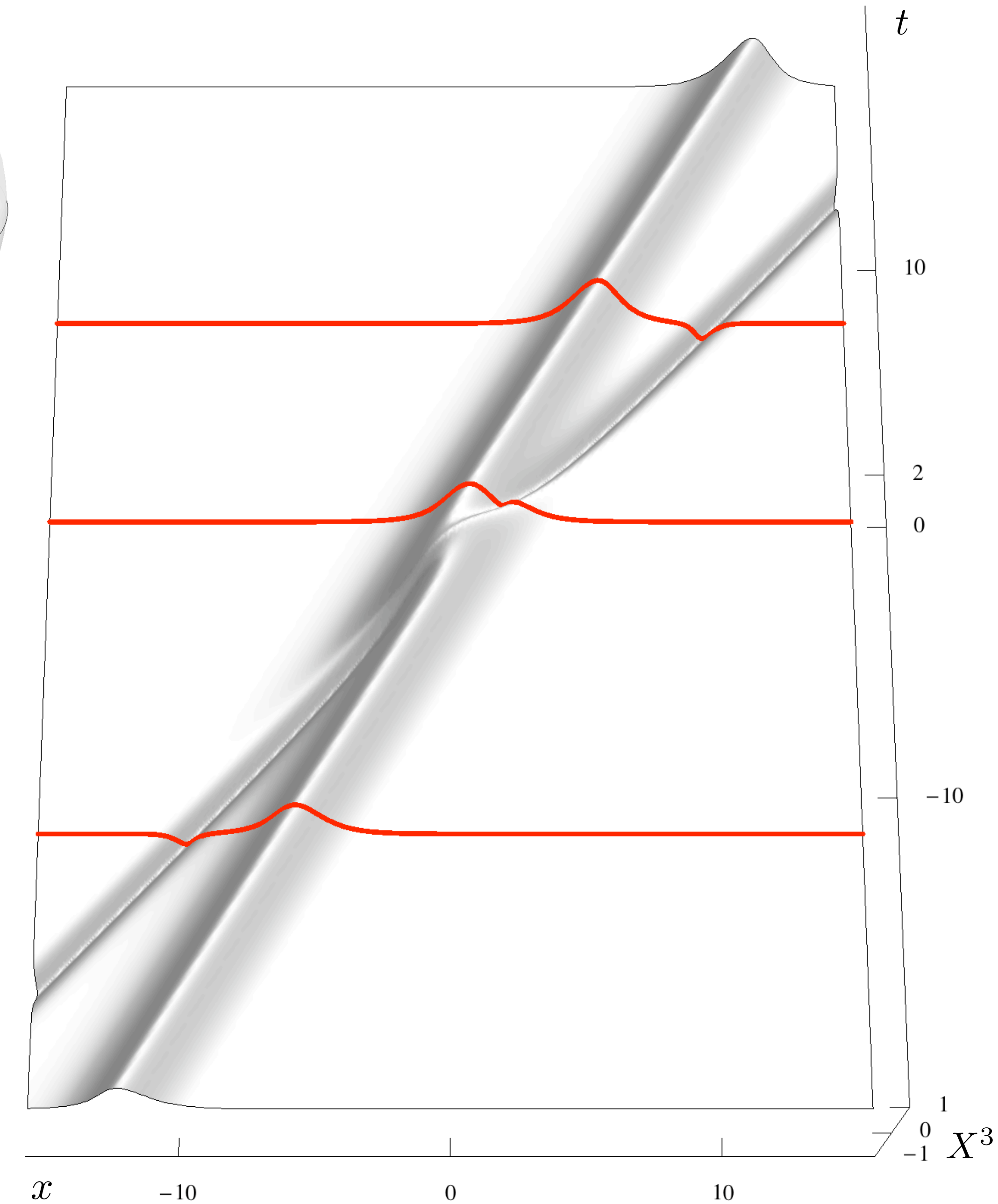
$t = 10$



$t = 2$



$t = -10$



Exact solution:

$$X^1 + iX^2 = e^{it}\left(1 + \frac{A + iB}{D}\right)$$

$$X^3 = (v_1 - v_2)\,\frac{\sqrt{1 - v_1^2}\cosh u_2 - \sqrt{1 - v_2^2}\cosh u_1}{D}$$

$$u_1 = \frac{x - v_1 t}{\sqrt{1 - v_1^2}} \quad \text{and} \quad u_2 = \frac{x - v_2 t}{\sqrt{1 - v_2^2}}$$

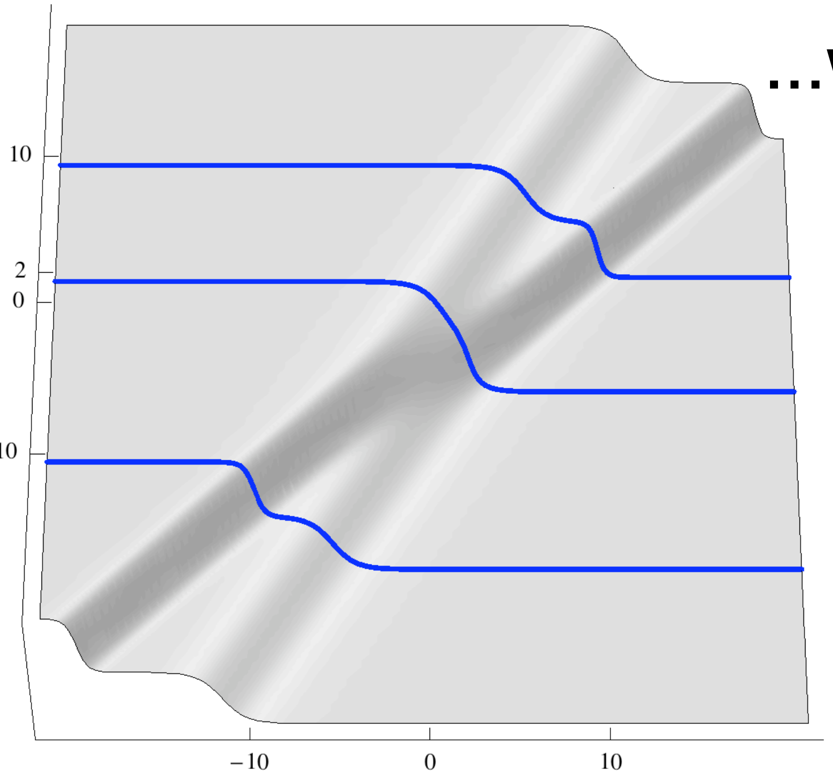
$$A = (v_1 - v_2)^2 \cosh u_1 \cosh u_2$$

$$B = (v_1 - v_2)\left(\sqrt{1 - v_1^2}\sinh u_1 \cosh u_2 - \sqrt{1 - v_2^2}\cosh u_1 \sinh u_2\right)$$

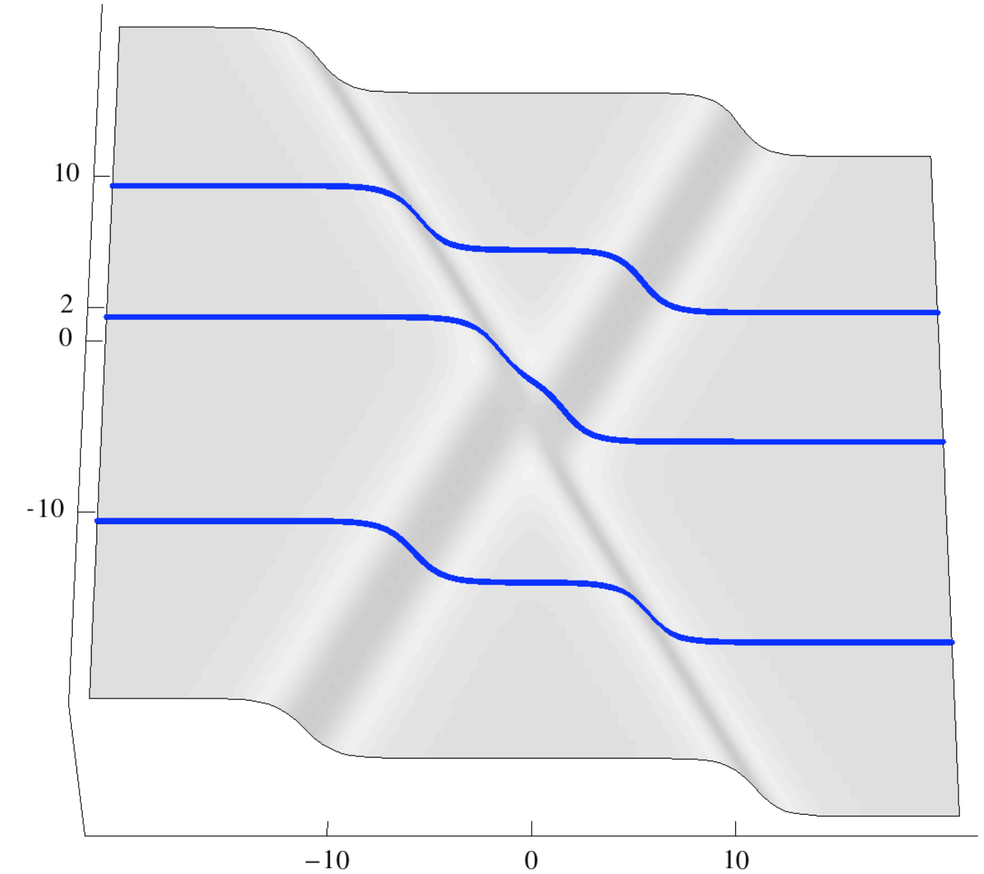
$$D = \sqrt{1 - v_1^2}\sqrt{1 - v_2^2}\,(1 + \sinh u_1 \sinh u_2) - (1 - v_1 v_2)\cosh u_1 \cosh u_2 .$$

[Spradlin & Volovich, 2006]

The Pohlmeyer map
sends that to this:



...which is a Lorentz
boost of this:



... which is very simple: $\tan\frac{\alpha}{4} = \frac{\cosh(\gamma v t)}{v\ \sinh(\gamma x)}$

We can read off the time delay $\Delta t_{v,-v} = 2\frac{\sqrt{1-v^2}}{v}\log v$

and then integrate it to find the phase shift: $\frac{\partial\delta_{1,2}}{\partial\mathcal{E}_1} = \Delta t_{v_1,v_2}$

i.e. $\delta_{1,2} = \int_0^{\mathcal{E}_1} \Delta t_{v(\mathcal{E}),v_2}\, d\mathcal{E}$

Here $\mathcal{E} = \Delta - J = \frac{\sqrt{\lambda}}{2}\sin\frac{p}{2}$.
while for sine-gordon you would use $E_{\text{s.g}}$

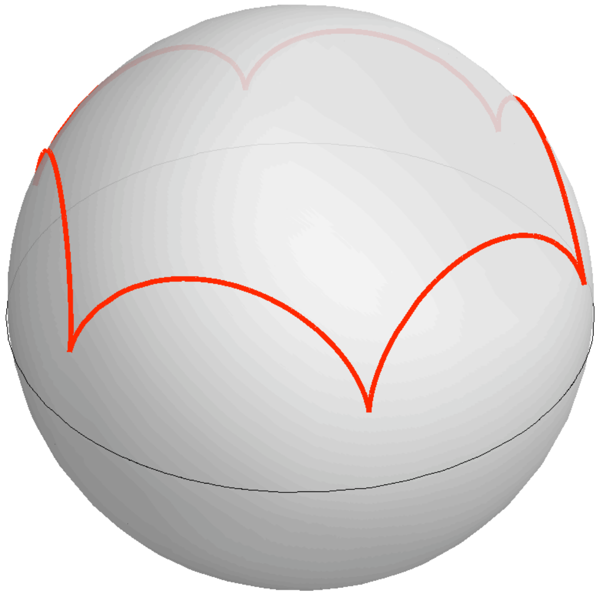
[Hoffman & Maldacena, 2006]

We've at $J = \infty$
$N = \infty$
$\lambda = \infty$ (or $\lambda \ll 1$)

Finite λ: Semiclassical quantisation... [Minahan 2006]
[Papathanasiou & Spradlin 2007]

Finite *J*:

- String solutions: explicitly find periodic solutions
- Sine-gordon solutions



[Arutynov, Frolov, Zamaklar, 2006]
[Astolfi, Forini, Grignani, Semenoff]
[Okamura & Suzuki 2006]

$$\delta\mathcal{E} \propto \sin^3\left(\frac{p}{2}\right) e^{-2\Delta/\mathcal{E}}$$

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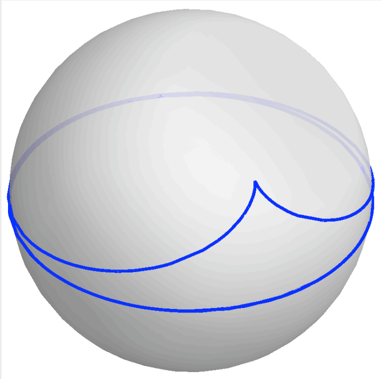
Semiclassical quantisation of the 'Single Spike'



[MCA & Aniceto, 2008]

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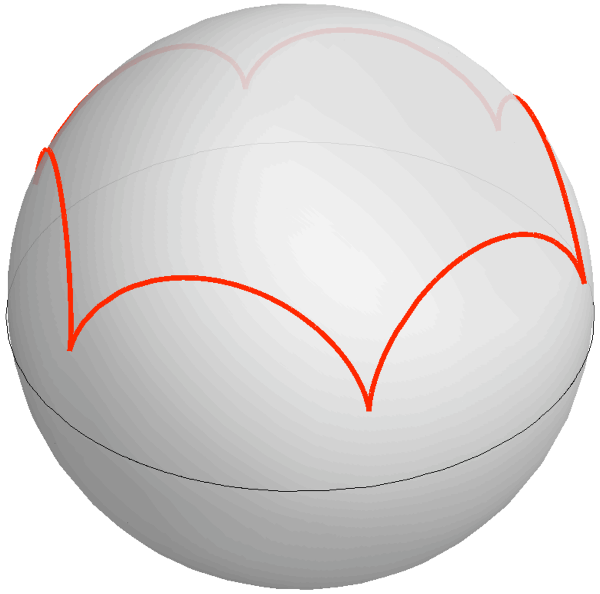

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Finite J:

- String solutions: explicitly find periodic solutions
- Sine-gordon solutions
- Algebraic Curves [Minahan & Sax, 2008] [Vicedo, 2008] [Lukowski & Sax, 2008]
- Lüsher formula [Janik & Lukowski]
- TBA [Gromov, Schafer-Nameki, Vieira]
- Konishi Operator [Keeler & Mann] [Fiamberti, Santambrogio, Sieg Zanon]

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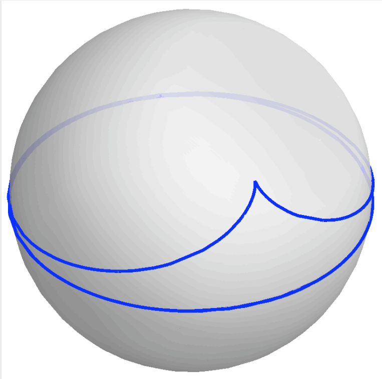

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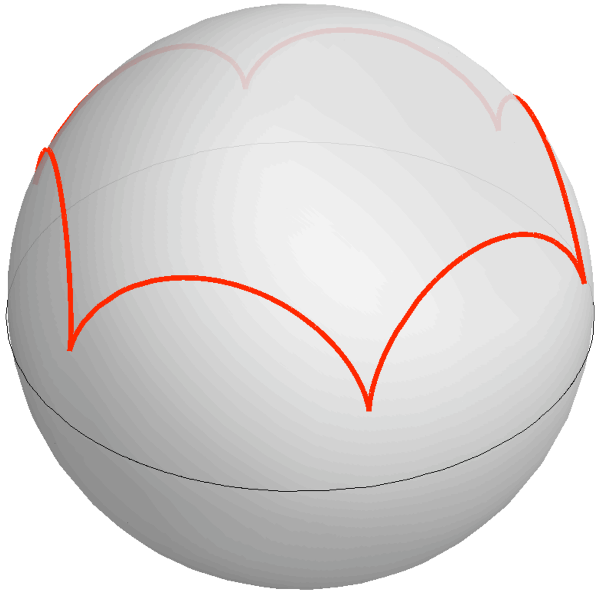

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3. ABJM
theory

The action is:

$$S = \frac{k}{4\pi} \int d^3x \ \mathrm{tr}\ \Big[\varepsilon^{\mu\nu\lambda} \Big(A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda \Big) + D_\mu Y_A^\dagger D^\mu Y^A$$
$$+ \frac{1}{12} Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + \frac{1}{12} Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C Y_A^\dagger - \frac{1}{2} Y^A Y_A^\dagger Y^B Y_C^\dagger Y^C Y_B^\dagger$$
$$+ \frac{1}{3} Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger + \text{fermions} \Big],$$

[Aharony, Bergman, Jafferis, Maldacena, 2008]

Symmetries

- Conformal theory in 2+1 dimensions: *so*(2,3)
- Four complex scalars in **4** of *su*(4) R-symmetry:

$$Y^A = (A_1, A_2, B_1^\dagger, B_2^\dagger)$$

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6 of *so*(6)

Adjoint $SU(N)$

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gauge indices: $Y_{a\bar{b}} Y^\dagger_{\bar{b}c}$

Spin-chain Picture

The $SU(2)$ x $SU(2)$ sector allows only $Y^1, Y^2, Y_3^\dagger, Y_4^\dagger$, and has vacuum:

$$\mathcal{O}_{\mathrm{vac}} = \mathrm{tr}\left(Y^1 Y_4^\dagger\right)^L$$

Here, at two loops, we get two decoupled Heisenberg spin chains.

$$\Delta - \frac{J}{2} = \sum_{i \text{ odd}} \mathcal{H}_{i,i+2} + \sum_{i \text{ even}} \mathcal{H}_{i,i+2}$$

[Minahan & Zarembo, 2008]
[Gaiotto, Giombi, Yin] [Bak & Rey]

(But weaker trace constraint: $\sum_{\mathrm{odd}} p_i + \sum_{\mathrm{even}} p_i = 0 \bmod 2\pi$)
Here $J(Y^1) = J(Y_4^\dagger) = \frac{1}{2}$, and we'll also need $Q(Y^2) = Q(Y_3^\dagger) = \frac{1}{2}$.

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String Dual

Membranes on $AdS_4 \times S^7/\mathbb{Z}_k$,
but when $N \to \infty$ with 't Hooft coupling $\lambda = \dfrac{N}{k}$ fixed,
we get IIA strings on $AdS_4 \times CP^3$,
of radius $R^2 = 2^{5/2}\pi\sqrt{\lambda}$.

Strings in CP^3

The metric is $ds^2 = \frac{R^2}{4} ds^2_{AdS} + R^2 ds^2_{CP} = R^2 \left(\frac{dy_\mu dy^\mu}{-4\mathbf{y}^2} + \frac{dz_i d\bar{z}_i}{|\mathbf{z}|^2} - \frac{|z_i d\bar{z}_i|^2}{|\mathbf{z}|^4} \right)$

CP^3 is defined as $\mathbb{C}^4$ with points $\mathbf{z} \sim re^{i\phi}\mathbf{z}$ identified.
Constrain length, and treat phase as a gauge freedom:

$$2\mathcal{L} = \frac{1}{4}\partial_a \mathbf{y} \cdot \partial^a \mathbf{y} - \Lambda(\mathbf{y}^2 + 1) + \overline{D_a \mathbf{z}} \cdot D^a \mathbf{z} - \Lambda'(\bar{\mathbf{z}} \cdot \mathbf{z} - 1)$$

where $D_a = \partial_a - A_a$, and the equations of motion read $A_a = \bar{\mathbf{z}} \cdot \partial_a \mathbf{z}$.

$$\partial_a \partial^a \mathbf{y} + (\partial_a \mathbf{y} \cdot \partial^a \mathbf{y})\mathbf{y} = 0,$$

$$D_a D^a \mathbf{z} + \left(\overline{D_a \mathbf{z}} \cdot D^a \mathbf{z} \right) \mathbf{z} = 0$$

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Now fix $y^{-1} + iy^0 = e^{2it}$ to concentrate on $\mathbb{R} \times CP^3$.
The $su(4)$ charges are defined

$$J[T^a] = 2\sqrt{2\lambda} \int dx \,\mathrm{Im}\left(\bar{\mathbf{z}} \cdot T^a D_t \mathbf{z} \right)$$

and we'll want:

$$J = J\left[\mathrm{diag}(1, 0, 0, -1)\right]$$

$$Q = J\left[\mathrm{diag}(0, 1, -1, 0)\right]$$

$$J_3 = J\left[\mathrm{diag}(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})\right]$$

Dispersion relation

$$\Delta - \frac{J}{2} = \sqrt{1 + h(\lambda)^2 \sin\left(\frac{p}{2}\right)}$$

with $h(\lambda) = \lambda$ from spin chain.

String States

The vacuum on the string side is $\mathbf{z} = \frac{1}{\sqrt{2}}(e^{it}, 0, 0, e^{-it})$.

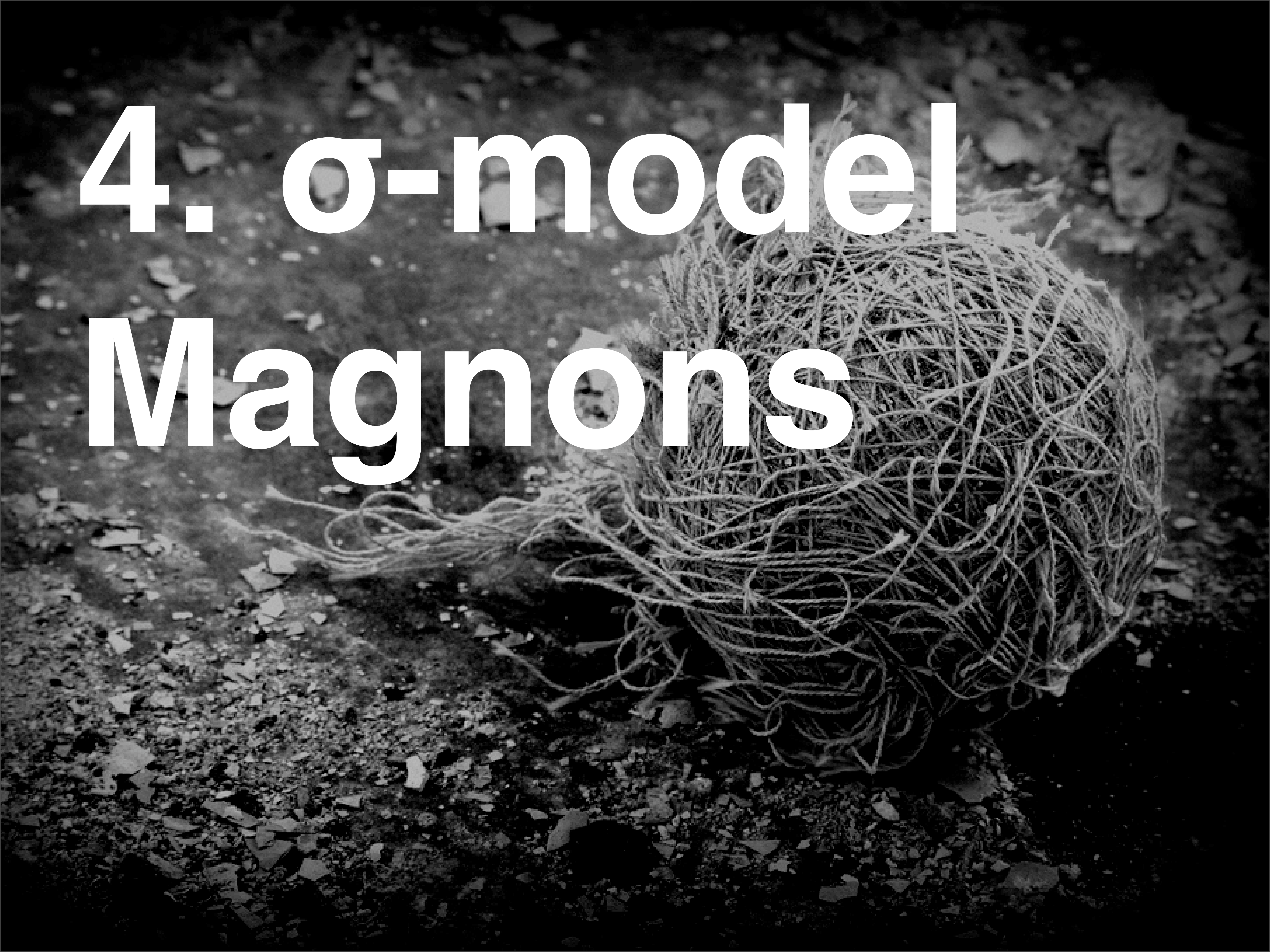
Study small fluctuations:
Penrose limit, or
Conformal gauge Δ-$J/2$

[Nishioka & Takayanagi, 2008]
[Gaiotto, Giombi, Yin] [Grignani, Harmark, Orselli]
[MCA & Aniceto, 2008]

Find 4 states $m = 1$ (all from CP^3)
and 4 states $m = 2$ (1 from CP^3, 3 from AdS_4)

And $h(\lambda) \sim \sqrt{\lambda}$.
(In previous case, this function is the same at in both limits.)

4. σ-model Magnons

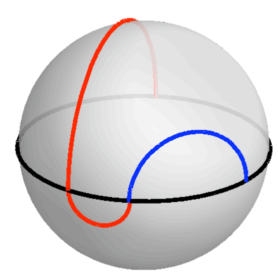


[Gaiotto, Giombi, Yin]

In fact there are two ways to embed the basic HM magnon:

1. Into $\boldsymbol{CP}^{1}$:

$$\mathbf{z}(x,t) = \begin{pmatrix} e^{\frac{i}{2}\phi_{\text{mag}}(2x,2t)}\sin\left(\frac{1}{2}\theta_{\text{mag}}(2x,2t)\right) \\ 0 \\ 0 \\ e^{-\frac{i}{2}\phi_{\text{mag}}(2x,2t)}\cos\left(\frac{1}{2}\theta_{\text{mag}}(2x,2t)\right) \end{pmatrix}$$



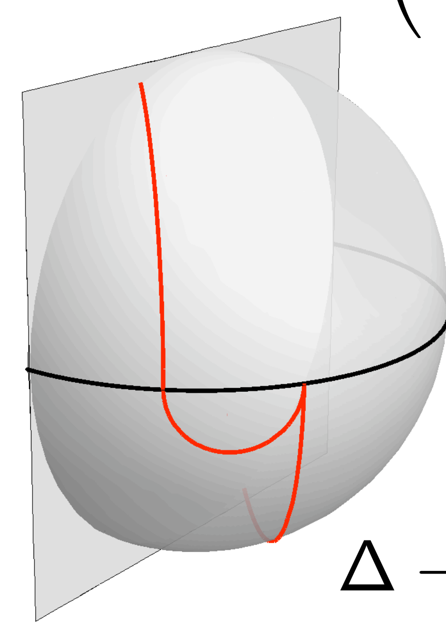
(to scale)

$$\Delta - \frac{J}{2} = \sqrt{2\lambda}\sin\left(\frac{p}{2}\right)$$

$$\sum_i p_i = 0 \bmod 2\pi$$

2. Into $\boldsymbol{RP}^{2}$:

$$\mathbf{z}(x,t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\phi_{\text{mag}}(x,t)}\sin\theta_{\text{mag}}(x,t) \\ \cos\theta_{\text{mag}}(x,t) \\ \cos\theta_{\text{mag}}(x,t) \\ e^{-i\phi_{\text{mag}}(x,t)}\sin\theta_{\text{mag}}(x,t) \end{pmatrix}$$



$$\Delta - \frac{J}{2} = 2\sqrt{2\lambda}\sin\left(\frac{p'}{2}\right)$$

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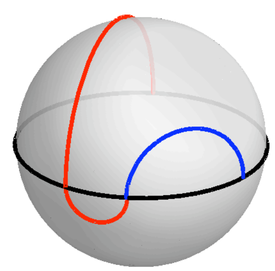

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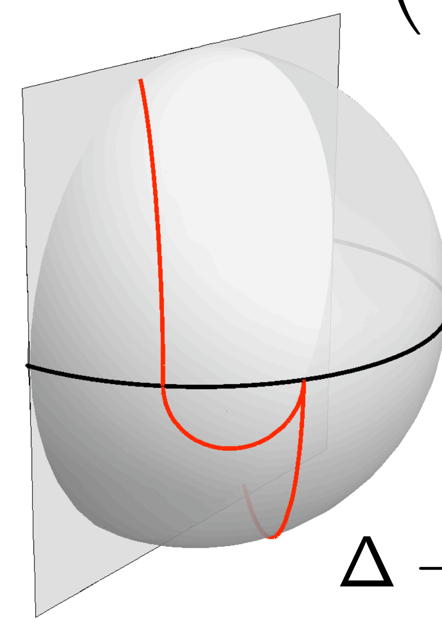

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Generalisations?

Tempting subspaces like $S^2 \times S^2$ [MCA & Aniceto]
but the equations of motion complain.

It is easy to embed Dorey's solution into $\boldsymbol{RP}^{\mathbf{3}}$:

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 8\lambda \sin\left(\frac{p'}{2}\right)}.$$

[Ahn, Bozhilov, Rashkov] [Ryang]

Can similarly embed finite-J magnons:

- First, AFZ form for two S^2-like subspaces:

CP^1: $$\Delta - \frac{J_1 - J_4}{2} = \sqrt{2\lambda}\sin\left(\frac{p}{2}\right)\left[1 - 4\sin^2\left(\frac{p}{2}\right)e^{-2\Delta\big/\sqrt{2\lambda}\sin(\frac{p}{2})} + \ldots\right]$$

[Lee, Panigrahi, Park]

RP^2: $$\Delta - \frac{J_1 - J_4}{2} = 2\sqrt{2\lambda}\sin\left(\frac{p}{2}\right)\left[1 - 4\sin^2\left(\frac{p}{2}\right)e^{-2\Delta\big/2\sqrt{2\lambda}\sin(\frac{p}{2})} + \ldots\right]$$

[Grignani, Harmark, Orselli, Semenoff]

- Second, embed the S^3 result of [Hatsuda & Suzuki, 2008] into RP^3:

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 8\lambda\sin^2\left(\frac{p'}{2}\right)} - 32\lambda\cos(2\phi)\frac{1}{\mathcal{E}}\sin^4\left(\frac{p'}{2}\right)e^{-\Delta\mathcal{E}\big/2S}$$

where $S = \dfrac{Q^2}{16\sin^2(\frac{p'}{2})} + 2\lambda\sin^2\left(\dfrac{p'}{2}\right)$

The factor $\cos(2\phi)$ describes a choice of relative orientation.

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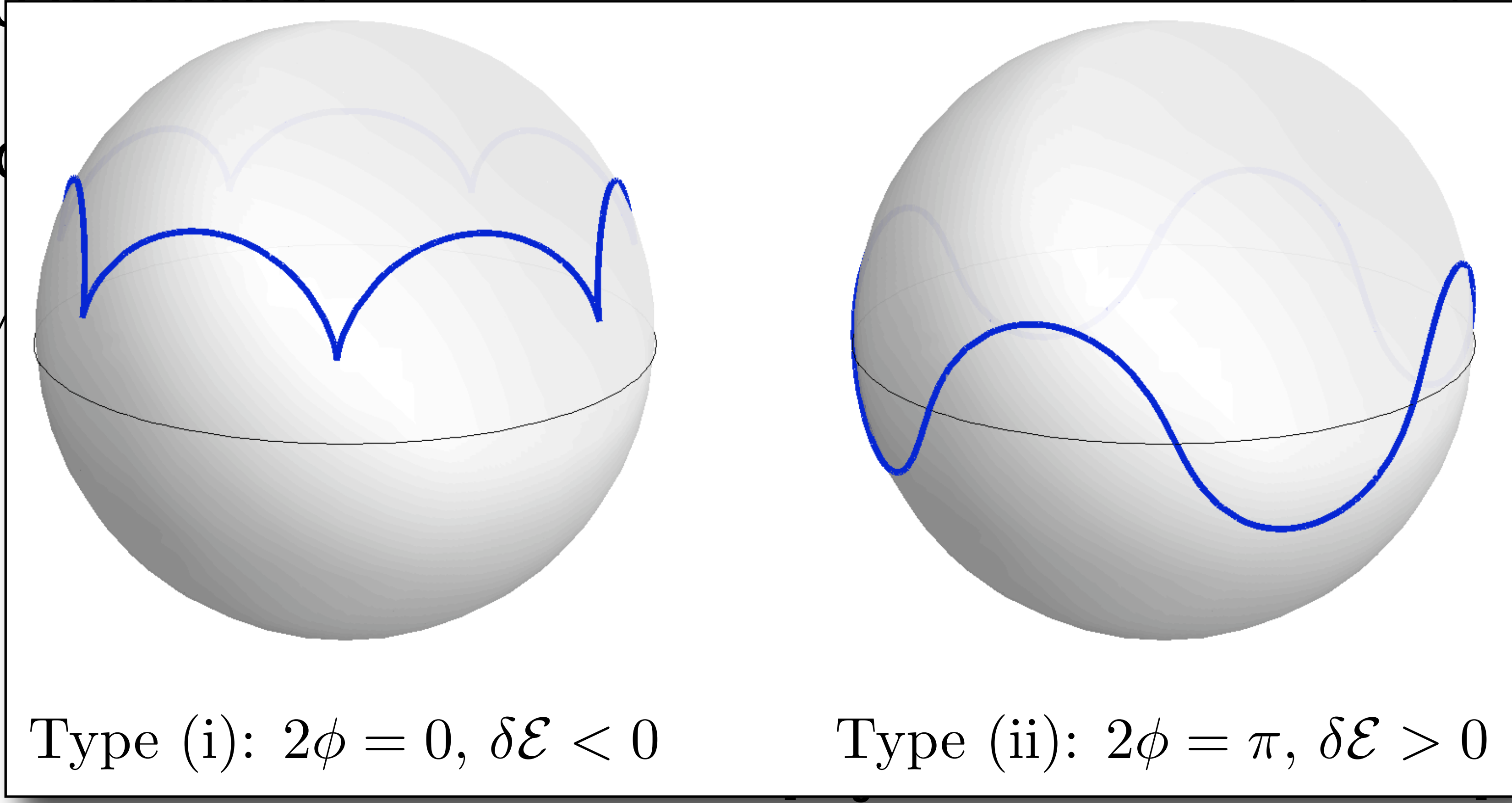

❖ First, AFZ form for tw

CP^1: $\Delta - \frac{J_1 - J_4}{2} = \sqrt{}$

RP^2: $\Delta - \frac{J_1 - J_4}{2} = 2$

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The factor $\cos(2\phi)$ describes a choice of relative orientation.

[MCA, Aniceto & Sax, 2009]

Dyonic CP2 solution

$$ds^2 = \frac{1}{4}\sin^2\xi \left[d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2 \;+ \cos^2\xi\,(d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2 \right] + d\xi^2$$

is the metric for this subspace, CP^2:

$$\mathbf{z} = \begin{pmatrix} \sin\xi\; \cos(\vartheta_2/2)\; e^{i\varphi_2/2} \\ \cos\xi\; e^{i\varphi_1/2} \\ 0 \\ \sin\xi\; \sin(\vartheta_2/2)\; e^{-i\varphi_2/2} \end{pmatrix}$$

Start with CP^1 magnon in $\xi = \pi/2$ space, with $p = \pi$.
Give it extra momentum along φ_1,

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Start with CP^1 magnon in $\xi = \pi/2$ space, with $p = \pi$.
Give it extra momentum along φ_1,
and deform ϑ_2 as for Dorey's solution:

$$\varphi_2 = 2t, \qquad \varphi_1 = -2\omega t,$$

$$\cos\vartheta_2 = \operatorname{sech}\left(\sqrt{1-\omega^2}2x\right), \qquad \xi = \frac{\pi}{2} - e(x).$$

Then solve for e(x)...

$$\cos^2 e(x) = \frac{1}{1+\omega\cos\vartheta_2}$$

Dyonic CP2 solution

[MCA, Aniceto & Sax, 2009]

$$ds^2 = \frac{1}{4}\sin^2\xi\left[d\vartheta_2^2 + \sin^2\vartheta_2\, d\varphi_2^2 + \cos^2\xi\,(d\varphi_1 - \cos\vartheta_2\, d\varphi_2)^2\right] + d\xi^2$$

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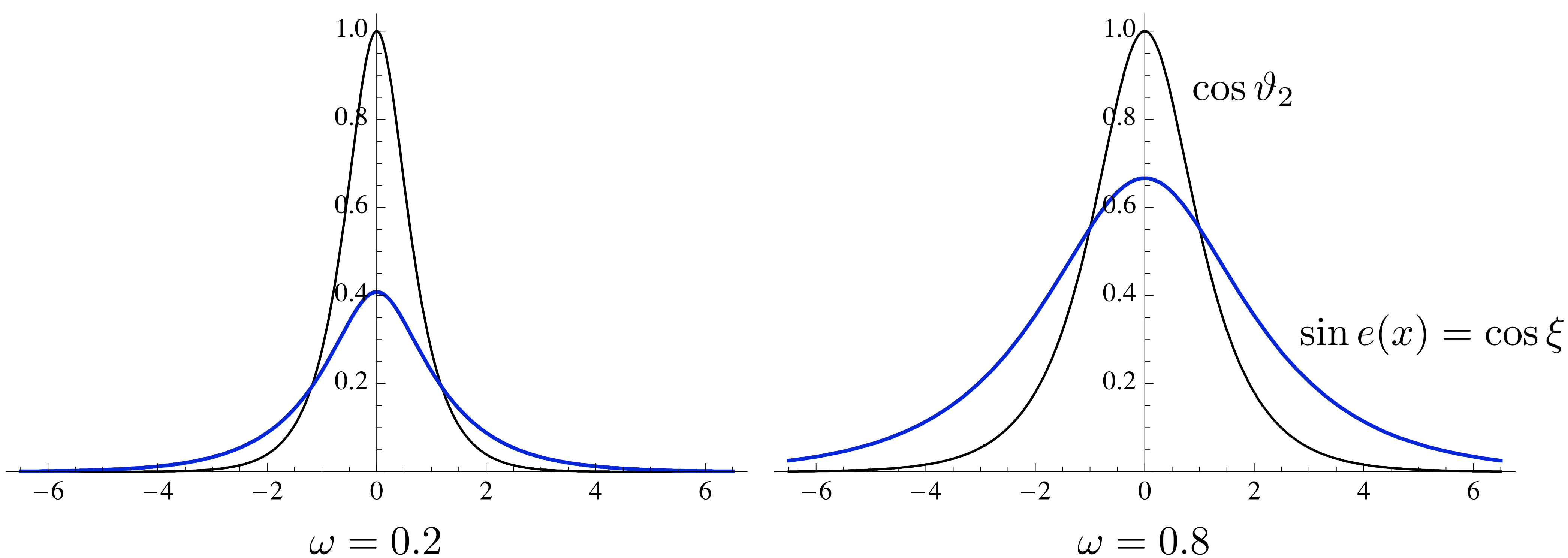

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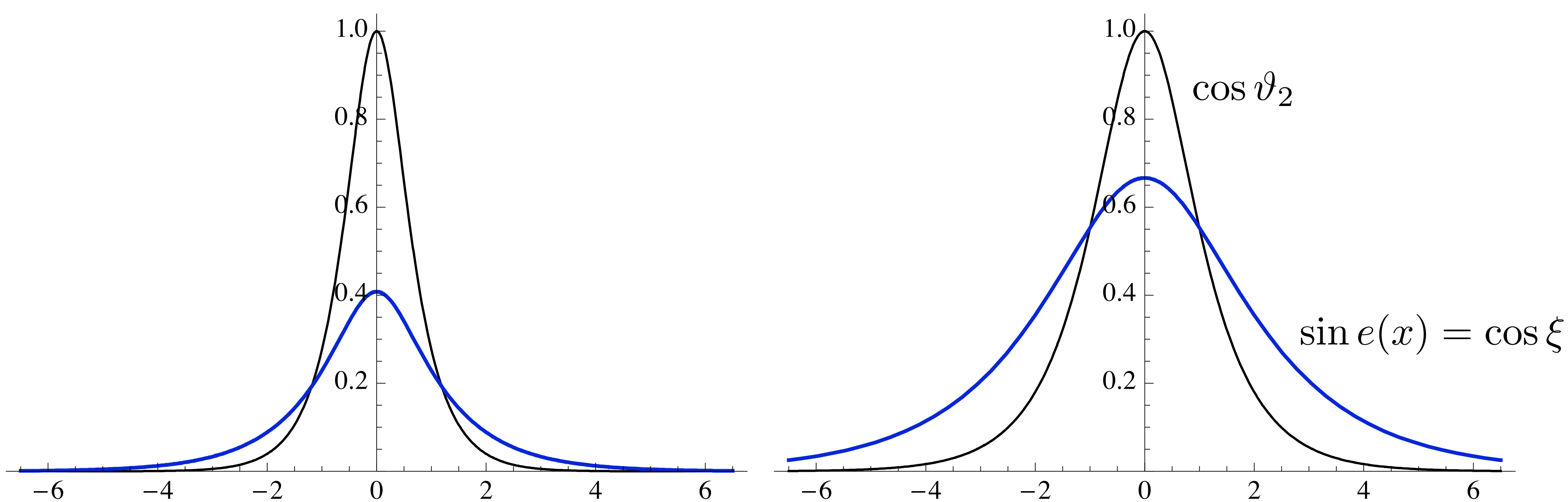
Note that two orientations are possible

Start with CP^1 magnon in $\xi = \pi/2$ space, with $p = \pi$.
Give it extra momentum along φ_1,



$\omega = 0.2$ $\qquad$ $\omega = 0.8$

Dressed Solution

Found by [Hollowood & Miramontes] + [Kalousios, Spradlin, Volovich] + [Suzuki] using the dressing method. The case $p = \pi$ looks like this:

$$\mathbf{z}' = N \begin{pmatrix} (1+r^2)\cos(t) + \cos\left(\frac{1-3r^2}{1+r^2}t\right) + r^2\cos\left(\frac{3-r^2}{1+r^2}t\right) + i(1-r^2)\sin(t)\sinh\left(\frac{4r}{1+r^2}x\right) \\ -(1+r^2)\sin(t) + \sin\left(\frac{1-3r^2}{1+r^2}t\right) - r^2\sin\left(\frac{3-r^2}{1+r^2}t\right) - i(1-r^2)\cos(t)\sinh\left(\frac{4r}{1+r^2}x\right) \\ 2(1-r^2)\left[\sin\left(\frac{1-r^2}{1+r^2}t\right)\sinh\left(\frac{2r}{1+r^2}x\right) - i\cos\left(\frac{1-r^2}{1+r^2}t\right)\cosh\left(\frac{2r}{1+r^2}x\right)\right] \\ 0 \end{pmatrix}$$

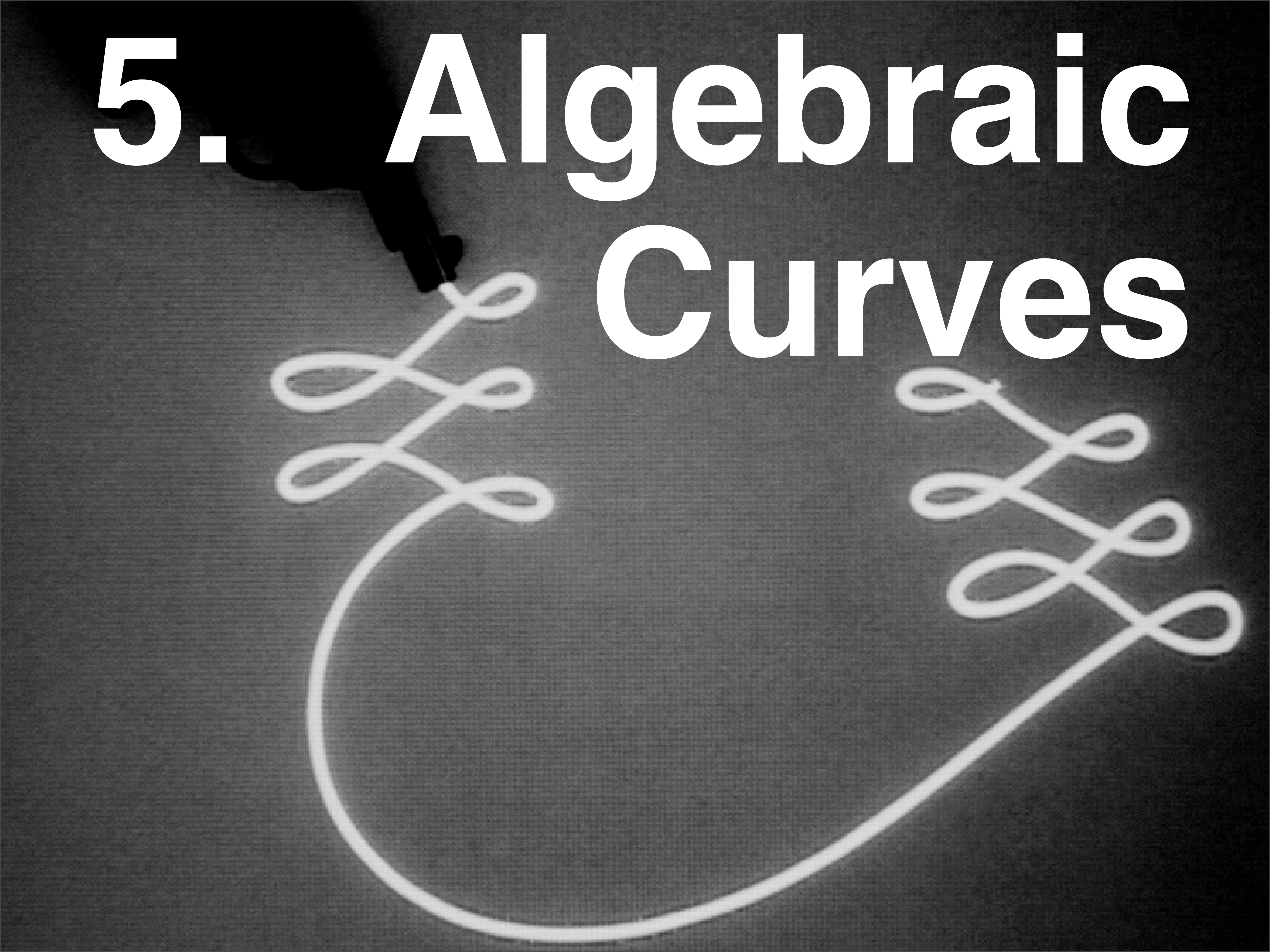
This has only one angular momentum:

$$J = 2\Delta - 4\sqrt{2\lambda}\frac{1+r^2}{2r}\sin\left(\frac{p'}{2}\right)$$

In the limit $r \to 1$, it becomes the RP^2 solution.

(This also lives in CP^2, so just call it ‘dressed’.)

5. Algebraic Curves



Getting there

Define two currents (left=right, vector representation):

$$(j_{AdS})_{ij,\mu} = 2\left(y_i \partial_\mu y_j - (\partial_\mu y_i) y_j\right)$$

$$(j_{CP})_{ij,\mu} = 2\left(z_i D_\mu z_j - (D_\mu z_i) z_j\right)$$

which are the blocks of $j = j_{AdS} \oplus j_{CP}$.

The Virasoro Constraints and equations of motion can now be written

$$dj + j \wedge j = 0 \qquad d * j = 0$$

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$$(j_{CP})_{ij,\mu} = 2\left(z_i D_\mu z_j - (D_\mu z_i) z_j\right)$$

which are the blocks of $j = j_{AdS} \oplus j_{CP}$.

The Virasoro Constraints and equations of motion can now be written

$$dj + j \wedge j = 0 \qquad\qquad d * j = 0$$

Define the Lax connection $J(x) = \frac{1}{1-x^2} j + \frac{x}{1-x^2} * j$

and using this, the monodromy matrix $\Omega(x) = P\, e^{\int d\sigma J_\sigma(x)}$.

The choice of path here determines the basis Ω is in, but its eigenvalues are invariant: $e^{i\tilde{p}_1}, e^{i\tilde{p}_2}, e^{i\tilde{p}_3}, e^{i\tilde{p}_4}$ and $e^{i\hat{p}_1}, e^{i\hat{p}_2}, e^{i\hat{p}_3}, e^{i\hat{p}_4}$.
We'll use

$$\{q_1, q_2, q_3, q_4, q_5\} = \frac{1}{2}\{\hat{p}_1 + \hat{p}_2, \hat{p}_1 - \hat{p}_2, \tilde{p}_1 + \tilde{p}_2, -\tilde{p}_2 - \tilde{p}_4, \tilde{p}_1 + \tilde{p}_4\}$$

[Gromov & Viera, 2008]

The Rules are:

1. Only 5 independent: $\{q_6, q_7, q_8, q_9, q_{10}\} = \{-q_5, -q_4, -q_3, -q_2, -q_1\}$.

2. Square-root branch cuts: $q_i^+(x) - q_j^-(x) = 2\pi n_{ij},\ x \in C_{ij}$.

3. Synchronised poles at $x = \pm 1$ with residue $\alpha/2$

4. Inversion symmetry:

$$q_1(\tfrac{1}{x}) = -q_2(x),\ q_3(\tfrac{1}{x}) = 2\pi m - q_4(x) \text{ and } q_5(\tfrac{1}{x}) = q_5(x).$$

Here $p = 2\pi m$ is the momentum

5. Asymptotic behaviour:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} = \frac{1}{2gx} \begin{pmatrix} \Delta + S \\ \Delta - S \\ J_1 \\ J_2 \\ J_3 \end{pmatrix} + \ldots \quad \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ b \end{matrix}}\right\} \text{AdS components} \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} \text{CP components} \end{array}$$

At large x, $\Omega(x) = 1 + \frac{1}{x} \int d\sigma j_\tau + \ldots$

(Here $4g = \sqrt{2\lambda}$)

The ansatz we'll use is this:

$$
\begin{aligned}
q_1(x) &= \frac{\alpha x}{x^2-1} \\
q_2(x) &= \frac{\alpha x}{x^2-1} \\
q_3(x) &= \frac{\alpha x}{x^2-1} \quad + G_u(0) - G_u(\tfrac{1}{x}) \quad + G_v(0) - G_v(\tfrac{1}{x}) \quad + G_r(x) - G_r(0) + G_r(\tfrac{1}{x}) \\
q_4(x) &= \frac{\alpha x}{x^2-1} \quad + G_u(x) \quad + G_v(x) \quad - G_r(x) + G_r(0) - G_r(\tfrac{1}{x}) \\
q_5(x) &= \quad G_u(x) - G_u(0) + G_u(\tfrac{1}{x}) \quad - G_v(x) + G_v(0) - G_v(\tfrac{1}{x})
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Each of G_u G_v G_r satisfies inversion symmetry, but only the first two contribute to $p = 2\pi m = q_3\left(\frac{1}{x}\right) + q_4(x)$.

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The labels are from $su(4)$ charges:

$$
\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} L - M_r \\ L + M_r - M_u - M_v \\ M_u - M_v \end{pmatrix}
$$

And $J = J_1 + J_2$, $Q = J_1 - J_2$.

Shenderovich's Giant Magnons -- insert $G_{\mathrm{mag}}(x) = -i \log\left(\frac{x - X^+}{x - X^-}\right)$

- **Small Magnon:** set $G_v(x) = G_{\mathrm{mag}}(x)$
  and expand things, read off:

$$p = -i \log \frac{X^+}{X^-}, \qquad Q = -i2g\left(X^+ - X^- + \frac{1}{X^+} - \frac{1}{X^-}\right)$$

$$J = 2\Delta + i2g\left(X^+ - X^- - \frac{1}{X^+} + \frac{1}{X^-}\right), \qquad J_3 = Q.$$

  which give dispersion relation:

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q^2}{4} + 16g^2 \sin^2\left(\frac{p}{2}\right)}$$

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- **Big Magnon:** set instead $G_u(x) = G_v(x) = G_r(x) = G_{\text{mag}}(x)$

$$p = -2i\log\frac{X^+}{X^-}, \qquad Q = 0$$

$$J = 2\Delta + i4g\left(X^+ - X^- - \frac{1}{X^+} + \frac{1}{X^-}\right) \qquad J_3 = 0$$

  thus dispersion relation:

$$\Delta - \frac{J}{2} = \sqrt{Q_u^2 + 64g^2\sin^2\left(\frac{p}{4}\right)}$$

  This Q_u is just a parameter!

We'll also want

- **Pair of Small magnons**, u+v like this: $G_v(x) = G_u(x) = G_{\mathrm{mag}}(x)$ with same X^+

  All the charges add, so we get

$$\Delta - \frac{J}{2} = \sqrt{\frac{Q_u^2}{4} + 16g^2 \sin^2\left(\frac{p_u}{2}\right)} + \sqrt{\frac{Q_v^2}{4} + 16g^2 \sin^2\left(\frac{p_v}{2}\right)}$$
$$= \sqrt{\frac{Q^2}{4} + 64g^2 \sin^2\left(\frac{p}{4}\right)}$$

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$$= \sqrt{\frac{Q^2}{4} + 64g^2 \sin^2\left(\frac{p}{4}\right)}$$

The Matching Problem

In 2008, it looked like:

Pair	RP^3	A perfect match.
?	CP^1	Finite-J: AFZ-like
Small	?	Finite-J: zero?
Big	?	

[Luckowski & Sax, 2008]

(One could have noticed that the bound state of [Spradlin & Volovich 2006] has the Big magnon's charges, but the wrong $Q \to 0$ limit.)

But then three things happened:

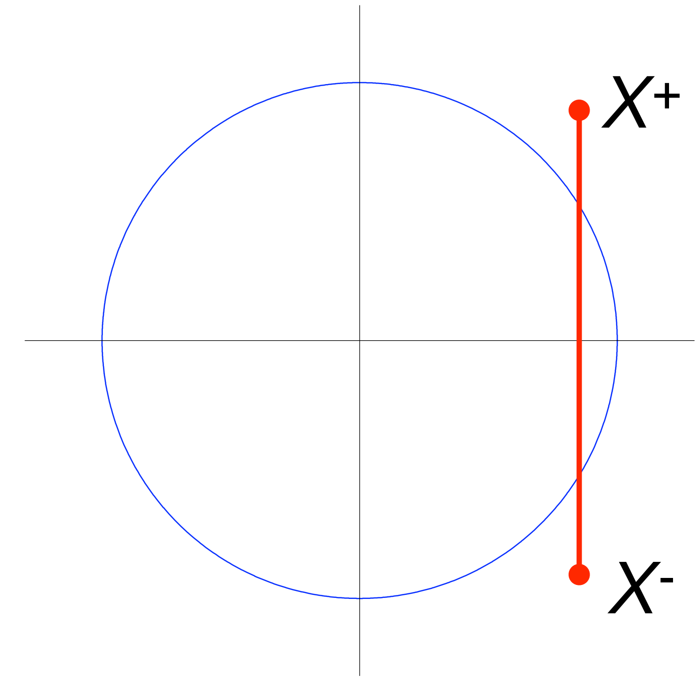
1. We fixed the finite-J calculation (= AFZ when not dyonic.)
2. The dressed solution appeared (Perfect!)
3. We found the CP^2 dyonic magnon (Comes in two polarisations.)

So the table becomes:

Pair	RP^3	non-dyonic is RP^2
Small	CP^2	non-dyonic is CP^1
Big	Dressed	non-‘dyonic’ is RP^2

Have already discussed 2 & 3, so we now turn to 1.

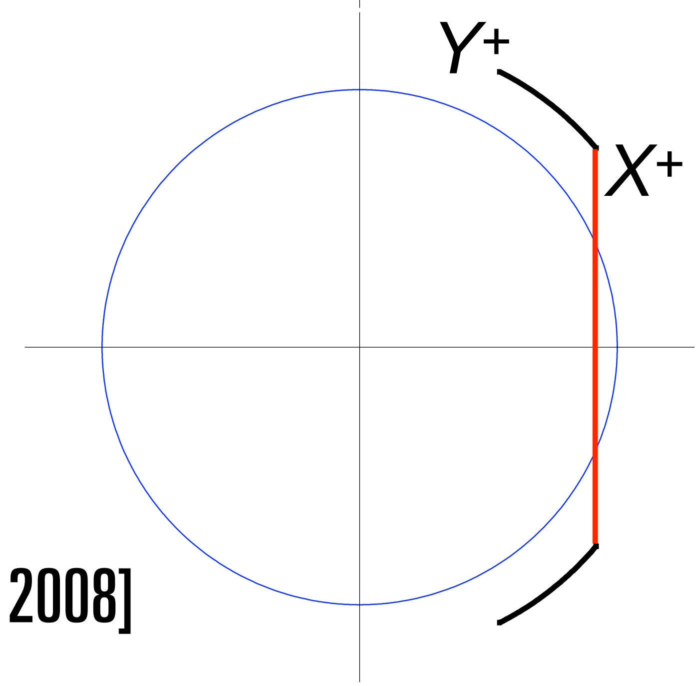
Finite-size corrections



The effect of J < Infinity is to replace:

$$G_{\mathrm{mag}}(x) = -i\log\left(\frac{x - X^+}{x - X^-}\right)$$

with this:

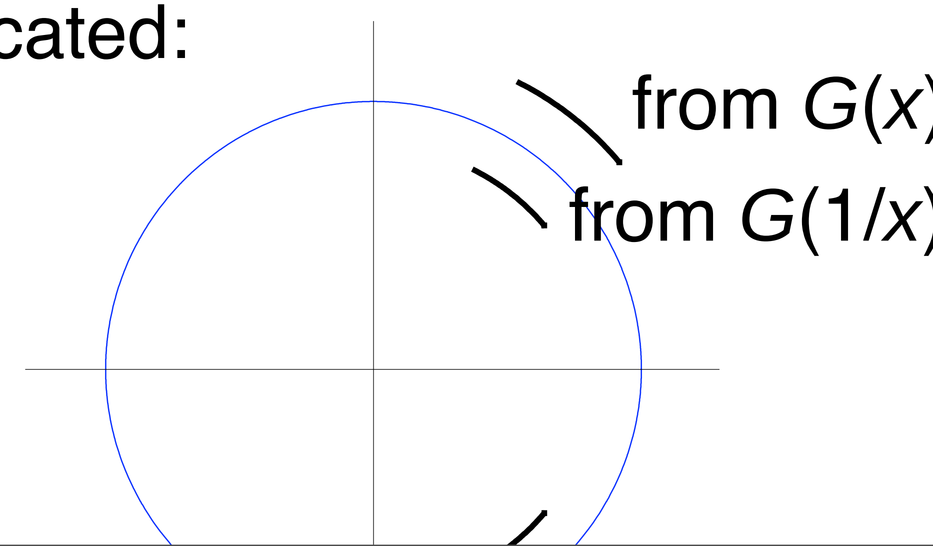


$$G_{\mathrm{finite}}(x) = -2i\log\left(\frac{\sqrt{x - X^+} + \sqrt{x - Y^+}}{\sqrt{x - X^-} + \sqrt{x - Y^-}}\right)$$

where $Y^{\pm} = X^{\pm}\left(1 \pm i\delta e^{\pm i\phi}\right)$ and $\delta \ll 1$.

[Minahan & Sax, 2008]

In q, the structure is more complicated:

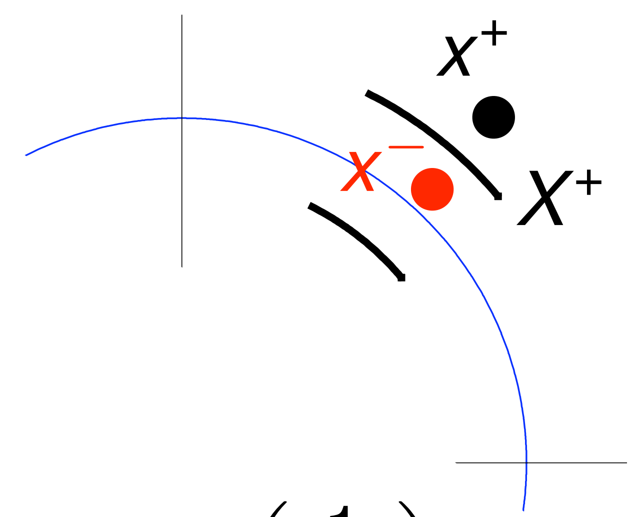


Finite-size Small Giant magnon

Defined by $G_u(x) = G_{\text{finite}}(x)$ which depends on $X^{\pm} = r\, e^{ip_0/2}$ and on δ, ϕ through $Y^{\pm} = X^{\pm}\left(1 \pm i\delta e^{\pm i\phi}\right)$.

1. Expand charges to find the correction:

$$\delta\mathcal{E} = \left(\Delta - \frac{J}{2}\right) - \sqrt{\frac{Q^2}{4} + 16 g^2 \sin^2\left(\frac{p}{2}\right)}$$
$$= -\delta^2 \frac{g}{4} \cos(2\phi) \frac{2r}{1+r^2} \sin\left(\frac{p}{2}\right) + o(\delta^3)$$



2. Impose the branch cut condition:

$$2\pi n = q_4(x^+) - q_6(x^-)$$
$$= \frac{2\alpha x}{x^2-1} + G^+_{\text{finite}}(X^+) + G^-_{\text{finite}}(X^+) - G_{\text{finite}}(0) + G_{\text{finite}}\left(\frac{1}{X^+}\right)$$

implies

$$\delta = \frac{8i\, e^{-ip/4} e^{i\pi n} e^{-i\phi} \sqrt{r^2-1}\, \sin(\frac{p}{2})}{\sqrt{e^{-ip/2} - r^2 e^{ip/2}}} \exp\left(\frac{i\Delta r/4g}{e^{-ip/2} - r^2 e^{ip/2}}\right)$$

3. Demand that delta be real:

$$n\pi - \frac{p}{4} - \phi - \frac{\Delta\, Q \cot(\frac{p}{2})}{4S(\frac{p}{2})} - \frac{1}{2}\arctan\left(\frac{2\mathcal{E}}{Q}\tan(\tfrac{p}{2})\right) = 0 \text{ or } \pi.$$

4. And get the answer:

$$\delta\mathcal{E} = -32g^2\cos(2\phi)\frac{Q}{\mathcal{E}\sqrt{S(\frac{p}{2})}}\sin^3\left(\frac{p}{2}\right)e^{-\Delta\mathcal{E}/S(\frac{p}{2})} + o(\delta^3)$$

[MCA, Aniceto & Sax, 2009]

where $S(\frac{p}{2}) = 4g^2\frac{(r^2-1)^2}{r^2} + 16g^2\sin^2\left(\frac{p}{2}\right)$

The non-dyonic limit (r → 1) of this is tricky.
We have implicitly assumed that $\delta \ll r - 1 \sim \sqrt{Q/g}$,
so can't simply set $Q = 0$.

Being careful, we can recover AFZ form $\delta\mathcal{E}_{r=1} = -16g\cos(2\phi)\sin^3\left(\frac{p}{2}\right)e^{-2\Delta/\mathcal{E}}$
and also closure condition $p = 0 \bmod 2\pi$

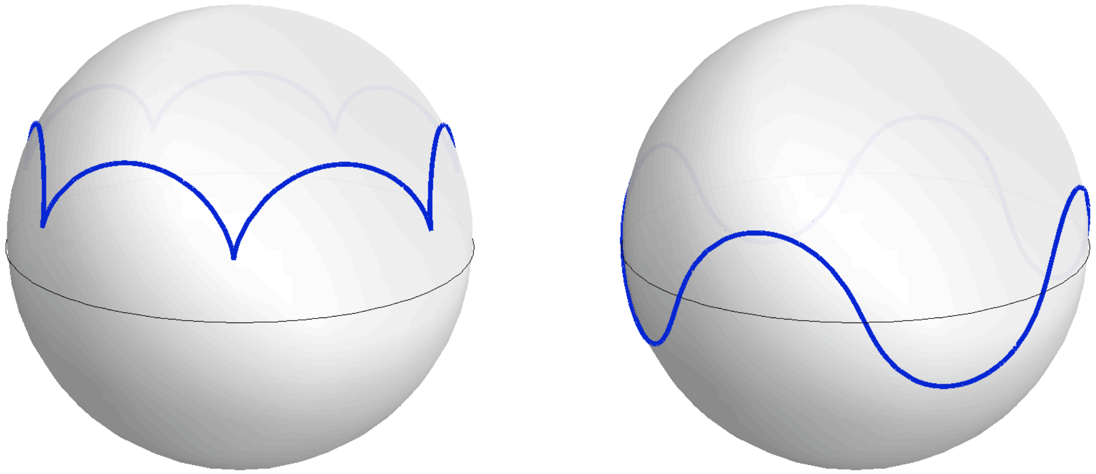
Pair of Small Magnons

(non-dyonic: [Lukowski & Sax, 2008])

We recover the S^3 answer: $\delta\mathcal{E} = -256 g^2 \cos(2\phi)\dfrac{1}{\mathcal{E}}\sin^4\left(\dfrac{p}{4}\right) e^{-\Delta\mathcal{E}/2S(\frac{p}{4})}$

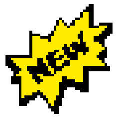
and the closure condition $p/2 = p' = n'\pi,\ n' \in \mathbb{Z}$,

[Hatsuda & Suzuki, 2008]
[Minahan & Sax, 2008]



Type (i): $2\phi = 0,\ \delta\mathcal{E} < 0$ Type (ii): $2\phi = \pi,\ \delta\mathcal{E} > 0$

Big Magnon

Here we get: 

$$\delta\mathcal{E} = -1024 g^2 \cos(2\phi)\frac{S(\frac{p}{4})}{\mathcal{E}Q_u^2}\sin^6\left(\frac{p}{4}\right) e^{-\Delta\mathcal{E}/S(\frac{p}{4})}$$

Non-‘dyonic’ limit similarly tricky,
but in the end Big = Pair = RP^2

[MCA, Aniceto & Sax, 2009]

Summary in the curves:

M_u, M_v, M_r	$[p_1, q, p_2]$	$\mathcal{E} = \Delta - \frac{J}{2}$	$\delta\mathcal{E}$ (finite J)	Q	J_3	String Solution
Vacuum						
$0, 0, 0$	$[L, 0, L]$	0	—	0	0	
Small giant magnon						
$1, 0, 0$	$[L-2, 1, L]$	$\sqrt{2\lambda}\sin(\frac{p}{2})$	$-4\mathcal{E}\sin^2\left(\frac{p}{2}\right) e^{-2\Delta/\mathcal{E}}$	1	1	$= CP^1$
Q, 0, 0	$[L-2Q, Q, L]$	$\sqrt{\frac{Q^2}{4} + 2\lambda\sin^2(\frac{p}{2})}$	$\propto Q/\mathcal{E}\sqrt{S}$,	Q	Q	$= CP^2$
... and similar with $u \leftrightarrow v$:						
0, Q, 0	$[L, Q, L-2Q]$	(same)	(same)	Q	$-Q$	
Big giant magnon						
$1, 1, 1$	$[L-1, 0, L-1]$	$2\sqrt{2\lambda}\sin(\frac{p}{4})$	$-4\mathcal{E}\sin^2\left(\frac{p}{4}\right) e^{-2\Delta/\mathcal{E}}$	0	0	$= RP^2$
Q_u, Q_u, Q_u	$[L-Q_u, 0, L-Q_u]$	$\sqrt{Q_u^2 + 8\lambda\sin^2(\frac{p}{4})}$	$\propto S/\mathcal{E}Q_u^2$,	0	0	= Dressed
Pair of small giant magnons						
$1, 1, 0$	$[L-2, 2, L-2]$	$2\sqrt{2\lambda}\sin(\frac{p}{4})$	$-4\mathcal{E}\sin^2(\frac{p}{4})e^{-2\Delta/\mathcal{E}}$	2	0	$= RP^2$
$\frac{Q}{2}, \frac{Q}{2}, 0$	$[L-Q, Q, L-Q]$	$\sqrt{\frac{Q^2}{4} + 8\lambda\sin^2(\frac{p}{4})}$	Like S^5 case	Q	0	$= RP^3$

The

END

Future

- Similar problem with limits in Lüsher calculation?
- Comparison with gauge side: big?
- Semiclassical corrections (add poles to curve).